

ON A MULTI-POINT SCHWARZ-PICK LEMMA

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ABSTRACT. We consider the multi-point Schwarz-Pick lemma and its associate functions due to Beardon-Minda and Baribeau-Rivard-Wegert. Basic properties of the associate functions are summarized. Then we observe that special cases of the multi-point Schwarz-Pick lemma give Schur's continued fraction algorithm and several inequalities for bounded analytic functions on the unit disk.

1. INTRODUCTION AND PRELIMINARIES

Many ways of applying the Schwarz lemma reveal deep properties of holomorphic mappings $f : \mathbb{D} \rightarrow \mathbb{D}$, where \mathbb{D} will denote the unit disk $\{z \in \mathbb{C} : |z| < 1\}$ throughout the present paper. For instance, the refined forms of the Schwarz Lemma due to Dieudonné and Rogosinski are explained in detail in [6]. More recently, a number of sharpened forms of the Schwarz or Schwarz-Pick Lemma have been obtained (see [2]-[5] and [8]). Among others, Beardon and Minda [4] presented an extension of the Schwarz-Pick Lemma which involves three points and yields known variations of the Schwarz-Pick Lemma in a unified way. Later on, Baribeau, Rivard and Wegert [2] generalized it to n points and applied it to Nevanlinna-Pick interpolation problem.

In this paper, we discuss the multi-point Schwarz-Pick Lemma by defining a set of holomorphic functions on \mathbb{D} associated with a sequence of given points in \mathbb{D} . We observe how our results are related with the Schur algorithm and show that they turn to coefficient estimates for a bounded analytic function on \mathbb{D} and there is a correlation between the coefficient estimates. Moreover, we obtain some applications of the results. We now start by recalling the Schwarz-Pick Lemma.

Lemma 1.1. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be holomorphic and fix $z_0 \in \mathbb{D}$. For any point $z \in \mathbb{D}$, the inequality*

$$\left| \frac{f(z) - f(z_0)}{1 - \overline{f(z_0)}f(z)} \right| \leq \left| \frac{z - z_0}{1 - \overline{z_0}z} \right| \quad (1.1)$$

holds if $z \neq z_0$ and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2} \quad (1.2)$$

if $z = z_0$. Equality holds for a point z precisely when f is a conformal automorphism of the unit disk \mathbb{D} .

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We denote by $\text{Aut}(\mathbb{D})$ the group of conformal automorphisms of \mathbb{D} . Note that $f : \mathbb{D} \rightarrow \mathbb{D}$ is in $\text{Aut}(\mathbb{D})$ if and only if $f(z) = (\alpha z + \beta)/(\bar{\beta}z + \bar{\alpha})$ for complex constants α and β with $|\alpha|^2 - |\beta|^2 = 1$. Furthermore, $H(\mathbb{D})$ will denote the set of holomorphic functions f on \mathbb{D} with $|f| \leq 1$. By the maximum principle, $f \in H(\mathbb{D})$ is a (unimodular) constant if f assumes a value in the boundary $\partial\mathbb{D}$ of \mathbb{D} . In other words, $|f| < 1$ in \mathbb{D} for $f \in H(\mathbb{D})$ unless f is a constant.

For $z, w \in \mathbb{D}$, let $[z, w]$ be defined by

$$[z, w] = \frac{z - w}{1 - \bar{w}z}. \quad (1.3)$$

Its modulus $|[z, w]|$ is called the pseudo-hyperbolic distance between z and w in \mathbb{D} [4]. It is convenient to memorize the fact that $w = [z, z_0]$ if and only if $z = [w, -z_0]$ for three points $z_0, z, w \in \mathbb{D}$. We extend the definition of $[z, w]$ by letting $[z, z] = 0$ for $z \in \partial\mathbb{D}$ so that $[f(z), f(w)]$ is defined whenever $f \in H(\mathbb{D})$ and $z, w \in \mathbb{D}$. The inequality (1.1) is now same as

$$|[f(z), f(z_0)]| \leq |[z, z_0]|. \quad (1.4)$$

The geometrical meaning of the Schwarz-Pick Lemma is that f is distance-decreasing with respect to the hyperbolic metric $\rho(z)|dz| = 2|dz|/(1 - |z|^2)$ of the unit disk. We denote by $d(z, w)$ the hyperbolic distance induced by ρ ; in other words,

$$d(z, w) = \log \frac{1 + |[z, w]|}{1 - |[z, w]|}.$$

The inequality (1.1) is equivalent to $d(f(z), f(z_0)) \leq d(z, z_0)$ for a holomorphic map $f : \mathbb{D} \rightarrow \mathbb{D}$ and $z, z_0 \in \mathbb{D}$.

Let us briefly recall the main idea of Beardon and Minda [4]. For this purpose, we introduce an operation for functions as follows. Let $f \in H(\mathbb{D})$ and $z_0 \in \mathbb{D}$. We define a holomorphic function $\Delta_{z_0}f$ on \mathbb{D} by

$$\Delta_{z_0}f(z) = \begin{cases} \frac{[f(z), f(z_0)]}{[z, z_0]} & \text{for } z \neq z_0, \\ \frac{(1 - |z_0|^2)f'(z_0)}{1 - |f(z_0)|^2} & \text{for } z = z_0. \end{cases} \quad (1.5)$$

The symbol $\Delta_{z_0}f$ is adopted in [2]. When it is convenient to regard $\Delta_{z_0}f(z)$ as a function of the two variables z and z_0 , we also write $\Delta_{z_0}f(z) = f_1(z; z_0)$. In [4], this quantity is called the ‘hyperbolic difference quotient’ of f , and the above notation is somewhat different from that of [4] for the purpose of introducing hyperbolic difference quotients of higher order. By the form of the definition, we have naturally the chain rule

$$\Delta_{z_0}(f \circ g) = (\Delta_{g(z_0)}f) \circ g \cdot \Delta_{z_0}g$$

for holomorphic maps $f, g : \mathbb{D} \rightarrow \mathbb{D}$ and $z_0 \in \mathbb{D}$ (cf. [4]). Since $\Delta_{z_0}T(z) = T'(z_0)/|T'(z_0)|$ for $T \in \text{Aut}(\mathbb{D})$, the following invariance property can easily be deduced.

Lemma 1.2. *Let $f : \mathbb{D} \rightarrow \mathbb{D}$ be a holomorphic map. For conformal automorphisms S and T of \mathbb{D} ,*

$$\Delta_{z_0}(S \circ f \circ T)(z) = \frac{S'(f(T(z_0)))}{|S'(f(T(z_0)))|} \cdot \frac{T'(z_0)}{|T'(z_0)|} \cdot \Delta_{T(z_0)}f(T(z))$$

for $z, z_0 \in \mathbb{D}$.

In particular, $|(S \circ f \circ T)_1(z; z_0)| = |f_1(T(z); T(z_0))|$ (cf. [4, Lemma 2.3]).

In terms of the hyperbolic difference quotient, the Schwarz-Pick Lemma is now rephrased as follows.

Lemma 1.3. *Let $f \in H(\mathbb{D})$. Then, for any pair of points $z, z_0 \in \mathbb{D}$,*

$$|\Delta_{z_0}f(z)| \equiv |f_1(z; z_0)| \leq 1. \quad (1.6)$$

Here, equality holds for a pair of points precisely when $f \in \text{Aut}(\mathbb{D})$.

Note that $\Delta_{z_0}f$ is a unimodular constant for any $z_0 \in \mathbb{D}$ when $f \in \text{Aut}(\mathbb{D})$ and that $|\Delta_{z_0}f| < 1$ on \mathbb{D} when $f \in H(\mathbb{D}) \setminus \text{Aut}(\mathbb{D})$.

It is crucial to note that, by the Schwarz-Pick Lemma, the function $\Delta_{z_0}f$ again belongs to $H(\mathbb{D})$ for $f \in H(\mathbb{D})$ and $z_0 \in \mathbb{D}$; in other words, Δ_{z_0} is an operator on $H(\mathbb{D})$ into itself. This observation leads to the following definition (cf. [2]): Let $f \in H(\mathbb{D})$. For a given (finite or infinite) sequence of points z_j ($j = 0, 1, \dots$) in \mathbb{D} , define $f_j(z; z_{j-1}, \dots, z_0)$ ($j = 0, 1, \dots$) by

$$f_j(z; z_{j-1}, \dots, z_0) = (\Delta_{z_{j-1}} \circ \dots \circ \Delta_{z_0})f(z).$$

Here, we understand that $f_0(z; -) = f(z)$ for $j = 0$. Note that this notation is consistent with the former definition of $f_1(z; z_0)$.

For brevity, we also write $f_j(z) = f_j(z; z_{j-1}, \dots, z_0)$ and $\gamma_j = f_j(z_j)$ for $j = 0, 1, 2, \dots$. We have then two possibilities:

- (i) $|\gamma_j| < 1$ for each j . Then $|f_j| < 1$ for each j . If f_j is constant for some j , then $f_k = \gamma_k = 0$ for $k > j$.
- (ii) There exists an integer n such that $|\gamma_0| < 1, |\gamma_1| < 1, \dots, |\gamma_{n-1}| < 1, |\gamma_n| = 1$. Then, $f_n = \gamma_n$ and f turns out to be a Blaschke product of degree n . Beardon and Minda [4] showed that this occurs only in this case. Here, we recall that a function f is called a (finite) Blaschke product of degree n if $f(z) = e^{i\theta} \prod_{j=1}^n [z, a_j]$ for $\theta \in \mathbb{R}$ and some points $a_1, \dots, a_n \in \mathbb{D}$. Note that $f_j = 0$ for $j > n$ in this case.

Through the above observation, for $f \in H(\mathbb{D})$, we see that $|f_n(z)| = 1$ for some $z \in \mathbb{D}$ if and only if f is a Blaschke product of degree n .

By repeated applications of the Schwarz-Pick lemma, we now have the following multi-point Schwarz-Pick Lemma due to Beardon-Minda [4] for $j = 2$ and Baribeau-Rivard-Wegert [2] for general j .

Theorem 1.4. *Let $f \in H(\mathbb{D})$ and z_0, z_1, \dots, z_j be a sequence of $j + 1$ points in \mathbb{D} . Then*

$$|f_j(z; z_{j-1}, \dots, z_0)| \leq 1, \quad z \in \mathbb{D}. \quad (1.7)$$

Equality holds for a point $z \in \mathbb{D}$ if and only if f is a Blaschke product of degree j . Moreover, if f is not a Blaschke product of degree $\leq j$,

$$d(f_j(z; z_{j-1}, \dots, z_0), f_j(z_j; z_{j-1}, \dots, z_0)) \leq d(z, z_j), \quad z \in \mathbb{D}. \quad (1.8)$$

Equality holds for a point $z \neq z_j$ precisely when f is a Blaschke product of degree $j + 1$.

It is shown in [4] that many known results in [6] and [8] can be derived based on the above theorem for $j = 2$; namely the ‘three-point’ Schwarz-Pick Lemma. In the present note, we give some consequences of n -point Schwarz-Pick Lemma. To this end, we also present a couple of basic properties of the quantities $f_j(z_j; z_{j-1}, \dots, z_0)$ for f and z_0, \dots, z_j in the next section. In Section 3, several interpretations and applications are given. Indeed, we will point out relations to the Schur algorithm and Peschl’s invariant derivatives, and give several concrete refinements of known results such as Yamashita’s inequality, Dieudonné’s lemma. For Dieudonné’s lemma [6], in addition to [4], see also [5]. We would like to remark that such refinements could be given, in principle, as much as we wish, with the expense of complication.

2. MAIN RESULTS

We first observe analyticity of the function $f_j(z; z_{j-1}, \dots, z_0)$ for $f \in H(\mathbb{D})$. This property guarantees existence of the limit of $f_j(z_j; z_{j-1}, \dots, z_0)$ as $z_k \rightarrow z_l$ for a pair of the variables z_k and z_l for instance, and allows us to change the order of limits.

Proposition 2.1. *Let $f \in H(\mathbb{D})$. Then for each $j \geq 0$, the function $f_j(z; z_{j-1}, \dots, z_0)$ is complex analytic in $z \in \mathbb{D}$ and real analytic in $z_0, \dots, z_{j-1} \in \mathbb{D}$.*

Proof. We show the assertion by induction on j . It is clear for $j = 0$. We assume that the assertion is valid up to j . By definition,

$$\begin{aligned} & f_{j+1}(z; z_j, \dots, z_0) \\ &= \frac{f_j(z; z_{j-1}, \dots, z_0) - f_j(z_j; z_{j-1}, \dots, z_0)}{z - z_j} \cdot \frac{1 - \overline{z_j}z}{1 - \overline{f_j(z_j; z_{j-1}, \dots, z_0)}f_j(z; z_{j-1}, \dots, z_0)}. \end{aligned}$$

There is nothing to show when $z \neq z_j$. Thus, it is enough to show analyticity at every point of the form $(z, z_j, z_{j-1}, \dots, z_0) = (a_j, a_j, a_{j-1}, \dots, a_0)$.

The second factor of the right-hand side in the above formula is clearly analytic in the sense of the assertion. Analyticity of the first factor follows from the next lemma by the interpretation $z_k = t_{2k+1} + it_{2k+2}$ for $k = 0, 1, \dots, j-1$ and $w = z_j$. \square

Lemma 2.2. *Suppose that a continuous function $F(z, t_1, \dots, t_n)$ is complex analytic in the complex variable z and real analytic in the real variables t_1, \dots, t_n . Then the difference quotient*

$$\frac{F(w, t_1, \dots, t_n) - F(z, t_1, \dots, t_n)}{w - z}$$

is complex analytic in z, w and real analytic in t_1, \dots, t_n .

Proof. For simplicity, we prove only in the case when $n = 1$. It is enough to see that $(F(w, t) - F(z, t))/(w - z)$ is complex analytic in $|z| < r/2, |w| < r/2$ and real analytic

in $|t| < \delta$ for small enough $r > 0$ and $\delta > 0$. We may assume that F is expanded in the form

$$F(z, t) = \sum_{j=0}^{\infty} A_j(z) t^j, \quad |z| < 2r, |t| < 2\delta$$

for some constants $r > 0$ and $\delta > 0$.

By convergence of the above series, there exists a constant $M > 0$ such that

$$|A_j(z)| \leq M\delta^{-j}, \quad |z| \leq r, \quad j = 0, 1, 2, \dots$$

Since $F(z, t)$ is complex analytic in z , Cauchy's integral formula yields the expression

$$\begin{aligned} \frac{F(w, t) - F(z, t)}{w - z} &= \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{F(\zeta, t)}{(\zeta - w)(\zeta - z)} d\zeta \\ &= \sum_{j=0}^{\infty} \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{A_j(\zeta)}{(\zeta - w)(\zeta - z)} d\zeta \cdot t^j \\ &\equiv \sum_{j=0}^{\infty} B_j(z, w) t^j \end{aligned}$$

for $|z| < r, |w| < r$. Here,

$$|B_j(z, w)| \leq \frac{4M}{r\delta^j}, \quad |z| < r/2, \quad |w| < r/2,$$

and thus the above series is indeed convergent in $|t| < \delta$. \square

The following generalization of Lemma 1.2 will be useful to reduce general questions to special ones.

Lemma 2.3. *Let $f \in H(\mathbb{D})$, $S, T \in \text{Aut}(\mathbb{D})$ and $z_0, \dots, z_{j-1}, z \in \mathbb{D}$. Then*

$$(S \circ f \circ T)_j(z; z_{j-1}, \dots, z_0) = \frac{S'(f(T(z_0)))}{|S'(f(T(z_0)))|} \cdot f_j(T(z); T(z_{j-1}), \dots, T(z_0)) \cdot \prod_{k=0}^{j-1} \frac{T'(z_k)}{|T'(z_k)|}.$$

In particular,

$$|(S \circ f \circ T)_j(z; z_{j-1}, \dots, z_0)| = |f_j(T(z); T(z_{j-1}), \dots, T(z_0))|.$$

Proof. We can easily verify the relation $[\zeta z, \zeta w] = \zeta[z, w]$ for $z, w \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}$. Therefore,

$$\Delta_{z_0}(\zeta f) = \zeta \Delta_{z_0} f$$

for $f \in H(\mathbb{D})$, $z_0 \in \mathbb{D}$ and $\zeta \in \partial\mathbb{D}$. For brevity, we put $\omega = S'(f(T(z_0)))/|S'(f(T(z_0)))|$, $\zeta_k = T'(z_k)/|T'(z_k)|$, $z' = T(z)$ and $z'_k = T(z_k)$. Note that $\omega, \zeta_k \in \partial\mathbb{D}$. By Lemma 1.2 together

with the above relation, we see

$$\begin{aligned}
(S \circ f \circ T)_2(z; z_1, z_0) &= \Delta_{z_1}(\Delta_{z_0}(S \circ f \circ T))(z) \\
&= \Delta_{z_1}(\omega \zeta_0(\Delta_{z'_0} f) \circ T)(z) \\
&= \omega \zeta_0 \Delta_{z_1}((\Delta_{z'_0} f) \circ T)(z) \\
&= \omega \zeta_0 \zeta_1 \Delta_{z'_1}(\Delta_{z'_0} f)(T(z)) \\
&= \omega \zeta_0 \zeta_1 f_2(z'; z'_1, z'_0).
\end{aligned}$$

In the same way, we can show the required relation for general j . \square

Let $f \in H(\mathbb{D})$ and a point $z \in \mathbb{D}$ be given. The most crude estimate for $f(z)$ is $|f(z)| \leq 1$. This, however, cannot be improved without any additional information about f . If we know about the value w_0 of f at a given point z_0 , then the estimate can be improved. For instance, when $f(0) = w_0$, we have the better estimate [9, p. 167]

$$\frac{|f(0)| - |z|}{1 - |z||f(0)|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |z||f(0)|}. \quad (2.1)$$

When more values of f (and possibly its derivatives) at points z_j for $j = 0, 1, 2, \dots$ are specified, we may improve the estimate more. Indeed, we are able to show the following.

Theorem 2.4. *Let a, z_0, \dots, z_n be given points in \mathbb{D} and put $\tau_j = [a, z_j]$ for $j = 0, 1, \dots, n$.*

- (i) *Suppose that $f \in H(\mathbb{D})$ is not a Blaschke product of degree at most n . Let $f_j(z) = f_j(z; z_{j-1}, \dots, z_0)$, $\gamma_j = f_j(z_j)$ for $j = 0, 1, \dots, n$. Define Möbius transformations A_j , $j = 0, 1, \dots, n$, by*

$$A_j(x) = \frac{\tau_j x + \gamma_j}{1 + \overline{\gamma_j} \tau_j x}.$$

Then $f(a) \in (A_0 \circ \dots \circ A_n)(\overline{\mathbb{D}})$. If furthermore f is not a Blaschke product of degree $n + 1$, $f(a) \in (A_0 \circ \dots \circ A_n)(\mathbb{D})$.

- (ii) *Conversely, suppose that points $\gamma_0, \gamma_1, \dots, \gamma_n \in \mathbb{D}$ are given. Let A_j be as above and choose an arbitrary point $b \in (A_0 \circ \dots \circ A_n)(\overline{\mathbb{D}})$. Then there exists a function $f \in H(\mathbb{D})$ with $f(a) = b$ such that $\gamma_j = f_j(z_j; z_{j-1}, \dots, z_0)$ for $j = 0, 1, \dots, n$.*

Proof. We first show (i). Let $w_j = f_j(a)$ for $j = 0, 1, \dots, n + 1$. Here, $f_{n+1}(z)$ is defined similarly. By assumption, $|\gamma_j| < 1$ and $|w_j| < 1$ for $j \leq n$. Also note that $|w_{n+1}| \leq 1$ and equality holds if and only if f is a Blaschke product of degree $n + 1$. Then, by definition,

$$w_{j+1} = \Delta_{z_j} f_j(a) = \frac{[w_j, \gamma_j]}{\tau_j},$$

and thus,

$$w_j = [\tau_j w_{j+1}, -\gamma_j] = A_j(w_{j+1}) \quad (2.2)$$

for $j = 0, 1, \dots, n$. Therefore, $w_0 = (A_0 \circ \dots \circ A_n)(w_{n+1})$ and (i) is proved.

We next show (ii). Set $c = (A_0 \circ \dots \circ A_n)^{-1}(b)$. Then, by assumption, $c \in \overline{\mathbb{D}}$. Let $w_{n+1} = c$ and define w_n, w_{n-1}, \dots, w_0 inductively by (2.2). Let f_{n+1} be any function in

$H(\mathbb{D})$ such that $f_{n+1}(a) = c$. For instance, f_{n+1} can be taken to be the constant function c . Then, define functions f_n, f_{n-1}, \dots, f_0 inductively by the formula

$$f_j(z) = [[z, z_j]f_{j+1}(z), -\gamma_j] = \frac{[z, z_j]f_{j+1}(z) + \gamma_j}{1 + \bar{\gamma}_j[z, z_j]f_{j+1}(z)}. \quad (2.3)$$

Then $f_j(z_j) = [0, -\gamma_j] = \gamma_j$ and therefore the relation $\Delta_{z_j} f_j = f_{j+1}$ holds. We now set $f = f_0$ so that $f_j(z; z_{j-1}, \dots, z_0) = f_j(z)$. In particular, $f_j(z_j; z_{j-1}, \dots, z_0) = f_j(z_j) = \gamma_j$. By (2.3), we have $f_j(a) = [\tau_j f_{j+1}(a), -\gamma_j] = A_j(f_{j+1}(a))$. Hence, $f(a) = f_0(a) = (A_0 \circ \dots \circ A_n)(f_{n+1}(a)) = b$. Thus, we have shown the existence of such an f . \square

In applications of the last theorem, it is convenient to note the following elementary fact: For a Möbius transformation $A(z) = \frac{az+b}{cz+d}$ with $|c| < |d|$,

$$w \in A(\overline{\mathbb{D}}) \Leftrightarrow \left| w - \frac{a\bar{c} - b\bar{d}}{|c|^2 - |d|^2} \right| \leq \left| \frac{ad - bc}{|c|^2 - |d|^2} \right|. \quad (2.4)$$

For instance, $f(a) \in A_0(\overline{\mathbb{D}})$ in the theorem means the inequality

$$\left| f(a) - \frac{(1 - |\tau_0|^2)\gamma_0}{1 - |\gamma_0\tau_0|^2} \right| \leq \frac{(1 - |\gamma_0|^2)|\tau_0|}{1 - |\gamma_0\tau_0|^2},$$

where $\tau_0 = [a, z_0]$ and $\gamma_0 = f(z_0)$.

As another application of the relation (2.2), we obtain the next result.

Theorem 2.5. *Let $f \in H(\mathbb{D})$ and $z_0 \in \mathbb{D}$. Then the double inequality*

$$\frac{||f(z_0)| - |[z, z_0]f_1(z; z_0)||}{1 - |[z, z_0]f(z_0)f_1(z; z_0)|} \leq |f(z)| \leq \frac{|f(z_0)| + |[z, z_0]f_1(z; z_0)|}{1 + |[z, z_0]f(z_0)f_1(z; z_0)|}$$

holds for $z \in \mathbb{D}$. Equality holds in the left-hand (right-hand) inequality if and only if either $f(z_0)f(z) = 0$ or else $\arg f(z) = \arg f(z_0) \pmod{2\pi}$ (respectively, $\arg f(z) = \arg f(z_0) + \pi \pmod{2\pi}$).

Proof. We first note the elementary inequalities (cf. [9, p. 167])

$$\frac{||b| - |a||}{1 - |ab|} = ||[b], [a]| \leq |[a, b]| \leq |[a], -[b]| = \frac{|a| + |b|}{1 + |ab|} \quad (2.5)$$

for $a, b \in \mathbb{D}$. Here, equality holds in the left-hand (right-hand) side if and only if either $ab = 0$ or else $(a/b) > 0$ (resp. $(a/b) < 0$). We now apply the above inequality to the choice $a = [f(z), f(z_0)] = [z, z_0]f_1(z; z_0) = \tau_0 w_1$ and $b = -f(z_0) = -\gamma_0$. Since $[a, b] = [\tau_0 w_1, -\gamma_0] = w_0 = f(z)$ by (2.2), we obtain the assertion. \square

By Lemma 1.3, we have the following.

Corollary 2.6. *Let $f \in H(\mathbb{D})$ and $z_0 \in \mathbb{D}$. Then the double inequality*

$$\max \left\{ \frac{|f(z_0)| - |[z, z_0]|}{1 - |[z, z_0]f(z_0)|}, 0 \right\} \leq |f(z)| \leq \frac{|f(z_0)| + |[z, z_0]|}{1 + |[z, z_0]f(z_0)|}$$

holds for $z \in \mathbb{D}$. When $z \neq z_0$, equality holds in the right-hand side only if $f \in \text{Aut}(\mathbb{D})$.

Note that the corollary reduces to (2.1) when $z_0 = 0$. Thus, Theorem 2.5 improves the inequality (2.1).

Theorem 2.4 gives precise information about the location of the value $f(z)$ but it might not be easy to use. We can extract more rough but convenient estimates for $|f(z)|$ as follows.

Theorem 2.7. *Let a, z_0, \dots, z_n be given points in \mathbb{D} . Suppose that $f \in H(\mathbb{D})$ is not a Blaschke product of degree at most n . Put $f_j(z) = f_j(z; z_{j-1}, \dots, z_0)$, $\gamma_j = f_j(z_j)$, $\tau_j = [a, z_j]$ for $j = 0, 1, \dots, n$. Then the chain of inequalities*

$$|f(a)| \leq (T_0 \circ \dots \circ T_n)(1) \leq \dots \leq (T_0 \circ T_1)(1) \leq T_0(1) \quad (2.6)$$

hold, where T_j are the functions defined by

$$T_j(x) = \frac{|\tau_j|x + |\gamma_j|}{1 + |\tau_j\gamma_j|x}.$$

Proof. Let $w_j = f_j(z)$ for $j = 0, 1, \dots, n+1$ as before. Note first that $T_j(x)$ is non-decreasing in $0 \leq x \leq 1$, that $T_j(1) \leq 1$, and that $|w_j| \leq 1$. Therefore, the inequalities

$$(T_0 \circ \dots \circ T_n)(1) \leq \dots \leq (T_0 \circ T_1)(1) \leq T_0(1)$$

clearly hold. Therefore, it is enough to show the inequality $|f(z)| \leq (T_0 \circ \dots \circ T_n)(1)$.

By the proof of Theorem 2.4, we have $f(a) = w_0 = (A_0 \circ \dots \circ A_n)(w_{n+1})$. Note that (2.5) implies $|A_j(w)| \leq T_j(|w|) \leq 1$ for $w \in \overline{\mathbb{D}}$. Therefore, we have

$$|f(a)| \leq T_0(|(A_1 \circ \dots \circ A_n)(w_{n+1})|) \leq \dots \leq (T_0 \circ \dots \circ T_n)(|w_{n+1}|) \leq (T_0 \circ \dots \circ T_n)(1),$$

as required. \square

The bound $T_0(1)$ in the last theorem is the same as in Corollary 2.6. The inequality for the next term $T_0(T_1(1))$ takes the form

$$|f(z)| \leq \frac{|f(z_0)| + \left| \frac{z-z_0}{1-\bar{z}_0 z} \right| \left| \frac{f(z_1)-f(z_0)}{1-f(z_0)f(z_1)} \right| + \left| \frac{z-z_1}{1-\bar{z}_1 z} \right| \left(|f(z_0)| \left| \frac{f(z_1)-f(z_0)}{1-f(z_0)f(z_1)} \right| + \left| \frac{z-z_0}{1-\bar{z}_0 z} \right| \right)}{1 + \left| \frac{z-z_0}{1-\bar{z}_0 z} \right| |f(z_0)| \left| \frac{f(z_1)-f(z_0)}{1-f(z_0)f(z_1)} \right| + \left| \frac{z-z_1}{1-\bar{z}_1 z} \right| \left(\left| \frac{f(z_1)-f(z_0)}{1-f(z_0)f(z_1)} \right| + |f(z_0)| \left| \frac{z-z_0}{1-\bar{z}_0 z} \right| \right)}.$$

Since $T_0(T_1(T_2(1)))$ is too complicated to write down, we restrict ourselves to the simple case when $z_0 = z_1 = \dots = z_n = 0$ so that $\tau_j = z$ for all j . For brevity, we write $c_j = |\gamma_j|$. Then the first three inequalities in Theorem 2.7 can be expressed by

$$\begin{aligned} |f(z)| &\leq \frac{c_0 + (c_1 + c_0 c_2 + c_0 c_1 c_2)|z| + (c_0 c_1 + c_2 + c_1 c_2)|z|^2 + |z|^3}{c_0 |z|^3 + (c_1 + c_0 c_2 + c_0 c_1 c_2)|z|^2 + (c_0 c_1 + c_2 + c_1 c_2)|z| + 1} \\ &\leq \frac{c_0 + (c_1 + c_0 c_1)|z| + |z|^2}{c_0 |z|^2 + (c_1 + c_0 c_1)|z| + 1} \\ &\leq \frac{c_0 + |z|}{1 + c_0 |z|}. \end{aligned}$$

Yamashita showed an inequality equivalent to the second one in [14, p. 313] and used it effectively to prove uniqueness of extremal functions in the norm estimates of starlike and

convex functions of order α in [15]. The above refinements could be used to improve the norm estimates.

As we saw before, the Schwarz-Pick lemma means the inequality $d(f(z), f(w)) \leq d(z, w)$ for a holomorphic map $f : \mathbb{D} \rightarrow \mathbb{D}$. This inequality can be refined by using the above argument.

Theorem 2.8. *Let $z, z_0, \dots, z_n \in \mathbb{D}$ and $f \in H(\mathbb{D})$. Suppose that f is not a Blaschke product of degree at most n . Let $R_0(x) = (1 + |\tau_0|x)/(1 - |\tau_0|x)$ and $T_j(x) = (|\tau_j|x + |\gamma_j|)/(1 + |\gamma_j|\tau_j|x)$, where $\tau_j = [z, z_j]$ and $\gamma_j = f_j(z_j; z_{j-1}, \dots, z_0)$. Furthermore set $R_n = R_0 \circ T_1 \circ T_2 \circ \dots \circ T_n$ for $n \geq 1$. Then,*

$$\exp(d(f(z), f(z_0))) \leq R_n(1) \leq R_{n-1}(1) \leq \dots \leq R_1(1) \leq R_0(1) = \exp(d(z, z_0)). \quad (2.7)$$

Proof. Define a Möbius transformation S by $S(x) = (1 + x)/(1 - x)$. Then we obtain $\exp(d(f(z), f(z_0))) = S(|[w_0, \gamma_0]|)$ and $(S^{-1} \circ R_0)(x) = |\tau_0|x$, where $w_0 = f(z)$. Thus we see that (2.7) is equivalent to

$$|\Delta_{z_0} f(z)| = \left| \frac{[w_0, \gamma_0]}{\tau_0} \right| \leq (T_1 \circ \dots \circ T_n)(1) \leq \dots \leq T_1(1) \leq 1,$$

which can be obtained by applying Theorem 2.7 to the function $\Delta_{z_0} f$ and the points z, z_1, \dots, z_n . \square

We consider the case when $z_0 = z_1 = z_2 = \dots$ and present explicit forms of $R_1(1)$ and $R_2(1)$. Put $t = |[z, z_0]|$ and $c_j = |f_j(z_0; z_0, \dots, z_0)|$. By a simple computation, we have

$$R_1(1) = \frac{1 + 2tc_1 + t^2}{1 - t^2}.$$

This was first obtained in [3]. The improvement of this bound in the next order is

$$R_2(1) = \frac{1 + t(c_1 + c_2 + c_1 c_2) + t^2(c_1 + c_2) + t^3}{1 + t(c_2 - c_1 + c_1 c_2) + t^2(c_1 - c_2) - t^3}.$$

Note that this is made possible by introducing the second order derivative of $f(z)$ through the term $c_2 = |f_2(z_0; z_0, z_0)|$.

3. INTERPRETATIONS OF THE RESULTS AND SOME APPLICATIONS

The most immediate and potentially important application of the multi-point Schwarz-Pick lemma is perhaps to the Nevanlinna-Pick interpolations as was developed by Baribeau, Rivard and Wegert [2]. Let us recall the Nevanlinna-Pick interpolation problem. Let z_0, z_1, \dots, z_n and w_0, w_1, \dots, w_n be given points in the unit disk \mathbb{D} . Here, for simplicity, we assume that z_0, \dots, z_n are distinct points. The Nevanlinna-Pick interpolation problem asks existence of a function $f \in H(\mathbb{D})$ such that

$$f(z_j) = w_j \quad \text{for } j = 0, 1, \dots, n. \quad (3.1)$$

The solvability of the Nevanlinna-Pick interpolation problem is characterized as positive semi-definiteness of the Hermitian form

$$Q(t_1, \dots, t_n) = \sum_{h,k=1}^n \frac{1 - w_h \bar{w}_k}{1 - z_h \bar{z}_k} t_h \bar{t}_k$$

(see for instance [1, §1.2]).

We notice that the parameters $\gamma_0, \gamma_1, \dots, \gamma_n$ are determined only by the data z_0, \dots, z_n and w_0, \dots, w_n when f is a solution to the problem (3.1). By Theorems 2.4 and 2.7, we have the following result.

Theorem 3.1. *Let z_0, z_1, \dots, z_n and w_0, w_1, \dots, w_n be given points in the unit disk \mathbb{D} with $z_j \neq z_k$ ($j \neq k$) and suppose that an analytic function $f : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ satisfies $f(z_j) = w_j$ ($j = 0, 1, \dots, n$). Let $\gamma_j = f_j(z_j; z_{j-1}, \dots, z_0)$ and $\tau_j = [z, z_j]$ for a fixed $z \in \mathbb{D}$. Then*

$$f(z) \in (A_0 \circ \dots \circ A_n)(\overline{\mathbb{D}}) \quad \text{and} \quad |f(z)| \leq (T_0 \circ \dots \circ T_n)(1),$$

where

$$A_j(x) = \frac{\tau_j x + \gamma_j}{1 + \bar{\gamma}_j \tau_j x}, \quad T_j(x) = \frac{|\tau_j| x + |\gamma_j|}{1 + |\tau_j \gamma_j| x}.$$

Remark 3.2. By the second part of Theorem 2.4, the set $(A_0 \circ \dots \circ A_n)(\overline{\mathbb{D}})$ is (so-called) the variability region of $f(z)$ for a given z concerning the solutions to the Nevanlinna-Pick interpolation problem in the theorem. Since the interpolation problem does not depend on the order of the data, this set remains unchanged if we change the order of the interpolation data $(z_0, w_0), (z_1, w_1), \dots, (z_n, w_n)$.

In the previous section, we often considered the case when $z_0 = z_1 = \dots$. This case is closely connected with the Schur algorithm and Peschl's invariant derivatives as we now see. Peschl's invariant derivatives $D_n f(z)$, $n = 1, 2, 3, \dots$, (with respect to the hyperbolic metric) are defined by the series expansion for $f \in H(\mathbb{D})$ [10] (see also [7] and [11]):

$$[f([z, -z_0]), f(z_0)] = \frac{f(\frac{z+z_0}{1+\bar{z}_0 z}) - f(z_0)}{1 - \bar{f}(z_0) f(\frac{z+z_0}{1+\bar{z}_0 z})} = \sum_{n=1}^{\infty} \frac{D_n f(z_0)}{n!} z^n, \quad z, z_0 \in \mathbb{D}.$$

Explicit forms of $D_n f(z)$, $n = 1, 2, 3$, are given by

$$D_1 f(z) = \frac{(1 - |z|^2) f'(z)}{1 - |f(z)|^2},$$

$$D_2 f(z) = \frac{(1 - |z|^2)^2}{1 - |f(z)|^2} \left[f''(z) - \frac{2\bar{z} f'(z)}{1 - |z|^2} + \frac{2\overline{f(z)} f'(z)^2}{1 - |f(z)|^2} \right],$$

and

$$D_3 f(z) = \frac{(1 - |z|^2)^3}{1 - |f(z)|^2} \left[f'''(z) - \frac{6\bar{z} f''(z)}{1 - |z|^2} + \frac{6\overline{f(z)} f'(z) f''(z)}{1 - |f(z)|^2} \right. \\ \left. + \frac{6\bar{z}^2 f'(z)}{(1 - |z|^2)^2} - \frac{12\bar{z} \overline{f(z)} f'(z)^2}{(1 - |z|^2)(1 - |f(z)|^2)} + \frac{6\overline{f(z)}^2 f'(z)^3}{(1 - |f(z)|^2)^2} \right].$$

Let us now recall the *Schur algorithm* [12] (see also [13]). Let $f \in H(\mathbb{D})$. Define functions f_0, f_1, f_2, \dots in $H(\mathbb{D})$ inductively by $f_0 = f$ and

$$f_{j+1}(z) = \frac{1}{z} \cdot \frac{f_j(z) - \gamma_j}{1 - \overline{\gamma_j} f_j(z)} = \frac{[f_j(z), \gamma_j]}{[z, 0]},$$

where $\gamma_j = f_j(0)$. The sequence $\{\gamma_j\}_{j=0}^\infty$ is called the *Schur parameter* of f . By construction, $f_j(z) = f_j(z; 0, \dots, 0)$ for $j = 0, 1, \dots$. Recall that either $|\gamma_j| < 1$ for all j or else $|\gamma_0| < 1, \dots, |\gamma_{n-1}| < 1, |\gamma_n| = 1, \gamma_{n+1} = \dots = 0$ for some $n \geq 0$. The latter case happens precisely when f is a Blaschke product of degree n .

We note that $D_1 f(z)$ is known as the hyperbolic derivative of f . We can easily see that $f_1(z; z) = D_1 f(z)$. The Schwarz-Pick lemma now implies $|D_1 f(z)| \leq 1$. What is the relation between $f_n(z; z, \dots, z)$ and $D_n f(z)$? The next result answers to it.

Proposition 3.3. *Let $f \in H(\mathbb{D})$ and $z_0 \in \mathbb{D}$. Define $g \in H(\mathbb{D})$ by $g(z) = [f([z, -z_0]), f(z_0)]$. Then $g^{(n)}(0) = D_n f(z_0)$ and $f_n(z_0; z_0, \dots, z_0) = \gamma_n$ for $n = 1, 2, \dots$, where $\{\gamma_n\}$ is the Schur parameter of g .*

Proof. The relations $g^{(n)}(0) = D_n f(z_0)$ immediately follow from the definition of $D_n f$. Define $S, T \in \text{Aut}(\mathbb{D})$ by $S(w) = (w - w_0)/(1 - \bar{w}_0 w)$ and $T(z) = (z + z_0)/(1 + \bar{z}_0 z)$, where $w_0 = f(z_0)$, so that $g = S \circ f \circ T$. Note that $S'(w_0) = 1/(1 - |w_0|^2) > 0$ and $T'(0) = 1 - |z_0|^2 > 0$. Then, by Lemma 2.3, we have

$$\gamma_n = g_n(0; 0, \dots, 0) = f_n(z_0; z_0, \dots, z_0).$$

□

When we express g by the series expansion $g(z) = \sum_{n=1}^\infty a_n z^n$, the first several γ_j 's are given by

$$\begin{aligned} \gamma_1 &= a_1, \\ \gamma_2 &= \frac{a_2}{1 - |a_1|^2}, \\ \gamma_3 &= \frac{a_3(1 - |a_1|^2) + \bar{a}_1 a_2^2}{(1 - |a_1|^2)^2 - |a_2|^2}, \\ \gamma_4 &= \frac{a_4[(1 - |a_1|^2)^2 - |a_2|^2] + 2\bar{a}_1 a_2 a_3(1 - |a_1|^2) + \bar{a}_1^2 a_2^3 + \bar{a}_2 a_3^2}{(1 - |a_1|^2)^3 - (1 - |a_1|^2)(|a_3|^2 + 2|a_2|^2) + |a_2|^4 - a_1 \bar{a}_2^2 a_3 - \bar{a}_1 a_2^2 \bar{a}_3}. \end{aligned}$$

By the multi-point Schwarz-Pick lemma (1.7), we have $|\gamma_n| = |g_n(0; 0, \dots, 0)| \leq 1$. Here, equality holds precisely if g (equivalently f) is a Blaschke product of degree n . Schur [12] indeed showed that the sequence of inequalities $|\gamma_n| \leq 1$ characterizes the boundedness of an analytic function f by 1 in modulus.

Noting the relation $a_n = g^{(n)}(0)/n! = D_n f(z_0)/n!$ by Proposition 3.3, we can rephrase the inequality $|\gamma_n| \leq 1$ in terms of Peschl's invariant derivatives. In particular, we obtain the following inequality due to Yamashita as the case when $n = 2$.

Proposition 3.4 (Yamashita [14, Theorem 2]). *Let $f \in H(\mathbb{D})$. Then,*

$$|D_2 f(z)| \leq 2(1 - |D_1 f(z)|^2), \quad z \in \mathbb{D}.$$

Equality holds for a point $z \in \mathbb{D}$ if and only if f is a Blaschke product of degree at most 2.

By the inequality $|\gamma_3| \leq 1$, we can similarly show the following.

Theorem 3.5. *Let $f \in H(\mathbb{D})$. Then, for $z \in \mathbb{D}$,*

$$\left| \frac{D_3 f(z)}{6} (1 - |D_1 f(z)|^2) + \overline{D_1 f(z)} \left(\frac{D_2 f(z)}{2} \right)^2 \right| + \left| \frac{D_2 f(z)}{2} \right|^2 \leq (1 - |D_1 f(z)|^2)^2,$$

where equality holds for a point $z \in \mathbb{D}$ if and only if f is a Blaschke product of degree at most 3.

We have considered the simplest case when $z_0 = z_1 = \dots = z_j$ so far. The second simplest case is perhaps when z_0, z_1, \dots, z_j consist of only two points. We start with $f_2(z; z, z_0)$. The inequality $|f_2(z; z, z_0)| \leq 1$ can be explicitly described in the following result.

Theorem 3.6 (Generalized Dieudonné's lemma). *Let f be an analytic function on \mathbb{D} with $|f| < 1$ and fix $z_0 \in \mathbb{D}$. Then, for any point $z \in \mathbb{D}$,*

$$\begin{aligned} & \left| f'(z) - \frac{f(z) - f(z_0)}{z - z_0} \cdot \frac{1 - \overline{f(z_0)}f(z)}{1 - |f(z_0)|^2} \cdot \frac{1 - |z_0|^2}{1 - \overline{z_0}z} \right| \\ & \leq \frac{1}{1 - |z|^2} \left(\frac{|1 - \overline{f(z_0)}f(z)|^2}{1 - |f(z_0)|^2} \cdot \left| \frac{z - z_0}{1 - \overline{z_0}z} \right| - \frac{|f(z) - f(z_0)|^2}{1 - |f(z_0)|^2} \cdot \left| \frac{1 - \overline{z_0}z}{z - z_0} \right| \right), \end{aligned} \quad (3.2)$$

where equality holds if and only if f is a Blaschke product of degree at most 2.

Proof. Let $g(z) = f_1(z; z_0)$. Then $f_2(z; z, z_0) = g_1(z; z)$. The inequality $|g_1(z; z)| = |D_1 g(z)| \leq 1$ is equivalent to

$$|g'(z)| \leq \frac{1 - |g(z)|^2}{1 - |z|^2}.$$

A straightforward calculation gives us the formula

$$g'(z) = \frac{f'(z)}{[z, z_0]} \cdot \frac{1 - |f(z_0)|^2}{(1 - \overline{f(z_0)}f(z))^2} - \frac{[f(z), f(z_0)]}{[z, z_0]^2} \cdot \frac{1 - |z_0|^2}{(1 - \overline{z_0}z)^2}.$$

It takes a little rearrangements for the required inequality. \square

Note that the inequality (3.2) is reduced to the original Dieudonné's lemma when $z_0 = f(z_0) = 0$:

$$|zf'(z) - f(z)| \leq \frac{|z|^2 - |f(z)|^2}{1 - |z|^2}.$$

Conversely, through elementary computations, it can be seen that the inequality (3.2) is obtained by applying the original Dieudonné's lemma to the function $h(\zeta) = [f([\zeta, -z_0]), f(z_0)]$ with the choice $\zeta = [z, z_0]$.

It turns out that the inequalities $|f_2(z; z_0, z)| \leq 1$ and $|f_2(z_0; z, z)| \leq 1$ are both equivalent to the inequality (3.2). Indeed, under the additional assumption that $z_0 = f(z_0) = 0$,

we have easily $|f_2(z; 0, z)| = |f_2(0; z, z)| = |[D_1 f(z), f(z)/z]/z| \leq 1$. If we set $w_0 = [D_1 f(z), f(z)/z]/z$, we have $D_1 f(z) = [zw_0, -f(z)/z]$ and the last inequality is equivalent to the assertion $D_1 f(z) \in A(\mathbb{D})$, where $A(w) = [zw, -f(z)/z]$. Now use (2.4) to see the equivalence with (3.2).

We next consider the case when $j = 3$ and z_0, z_1, z_2, z_3 consists of two points. By Lemma 1.2, we can assume that the two points are 0 and z (which are interchangeable) and that $f(0) = 0$. The inequality $|f_3(z; 0, 0, 0)| \leq 1$ means that the third function f_3 in the Schur algorithm has modulus at most 1. On the other hand, the inequality $|f_3(z; z, z, 0)| \leq 1$ is rearranged to the following, which can be regarded as Dieudonné's lemma of the second order.

Theorem 3.7. *Let $f \in H(\mathbb{D}) \setminus \text{Aut}(\mathbb{D})$ with $f(0) = 0$. Then*

$$\left| \frac{1}{2} z^2 f''(z) - \frac{zf'(z) - f(z)}{1 - |z|^2} + \frac{\overline{f(z)}(zf'(z) - f(z))^2}{|z|^2 - |f(z)|^2} \right| + \frac{|z||zf'(z) - f(z)|^2}{|z|^2 - |f(z)|^2} \leq \frac{|z|(|z|^2 - |f(z)|^2)}{(1 - |z|^2)^2}.$$

Proof. Let $g(z) = f(z)/z$. Then g is a holomorphic self-map of \mathbb{D} by assumption and $f_3(z; z, z, 0) = g_2(z; z, z)$. The inequality $|f_3(z; z, z, 0)| = |g_2(z; z, z)| \leq 1$ is thus equivalent to $\frac{1}{2}|D_2 g(z)| + |D_1 g(z)|^2 \leq 1$ (cf. Proposition 3.4). The last inequality is indeed equivalent to the inequality in question. \square

Finally, we consider the inequality $|f_3(z; z, 0, 0)| \leq 1$ under the condition $f(0) = 0$. Then we have the following inequality, which is another refinement of Dieudonné's lemma involving the term $f'(0)$.

Theorem 3.8. *Let $f \in H(\mathbb{D})$ with $f(0) = 0$. Then*

$$\left| f'(z)(1 - |f'(0)|^2) - \frac{2f(z)}{z} + \overline{f'(0)} \cdot \left(\frac{f(z)}{z} \right)^2 + f'(0) \right| \leq \frac{1}{1 - |z|^2} \left(\left| z - \overline{f'(0)}f(z) \right|^2 - \left| \frac{f(z)}{z} - f'(0) \right|^2 \right).$$

In particular, if $f'(0) = 0$ in addition,

$$\left| f'(z) - \frac{2f(z)}{z} \right| \leq \frac{|z|^4 - |f(z)|^2}{|z|^2(1 - |z|^2)}.$$

Proof. Since

$$f_3(z; z, 0, 0) = \frac{(1 - |z|^2)f_2'(z; 0, 0)}{1 - |f_2(z; 0, 0)|^2}$$

and

$$f_2(z; 0, 0) = \frac{1}{z} \cdot \frac{f(z) - zf'(0)}{z - \overline{f'(0)}f(z)},$$

we obtain the first inequality in the proposition by straightforward calculations. \square

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